MATH3210 - SPRING 2024 - SECTION 001

HOMEWORK 2

Problem 1. Consider the expression $x = .99999\ldots$, which we think of as a decimal expansion. That is, if $x_n = \sum_{k=1}^{n} 9 \cdot 10^{-k}$, we think of it as

$$x = \lim_{n\to\infty} .9999\ldots 9 = \lim_{n\to\infty} x_n.$$

Show that $x = 1$. [*Hint*: First show that $x_n + 10^{-n} = 1$ by induction]

Solution. We first show that $x_n + 10^{-n} = 1$ by induction. When $n = 1$, this follows because $x_1 = .9$ and $10^{-1} = .1$. Assume $x_n + 10^{-n} = 1$, and consider x_{n+1}. Then

$$\begin{aligned} x_{n+1} + 10^{-(n+1)} &= \left(\sum_{k=1}^{n+1} 9 \cdot 10^{-k}\right) + 10^{-(n+1)} \\ &= x_n + 9 \cdot 10^{-(n+1)} + 10^{-(n+1)} = x_n + 10 \cdot 10^{-(n+1)} = x_n + 10^{-n} = 1. \end{aligned}$$

Therefore, by induction $x_n + 10^{-n} = 1$ for all n.

Next, we show that $\lim_{n\to\infty} 10^{-n} = 0$. Indeed, first note that $n \leq 10^n$ for all $n \in \mathbb{N}$. We show this again by induction. Since $1 \leq 10$, this is clearly true for $n = 1$. Then if $n \leq 10^n$, $n + 1 \leq 10^n + 1 \leq 10^n + 9 \cdot 10^n = 10^{n+1}$. Now, since $n \leq 10^n$,

$$0 \leq 10^{-n} \leq 1/n.$$

Since $0 \to 0$ and $1/n \to 0$, it follows that $10^{-n} \to 0$ by the squeeze theorem. Then, by the limit arithmetic theorem, it follows that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 1 - 10^{-n} = 1 - 0 = 1.$$

□

Remark 1. It would be okay to omit the blue section in a solution, but I include it for clarity and extra exposition.

Problem 2. Consider the following property for a set $A \subset \mathbb{R}$.

(N) If $x \in A$, then there exists some $\varepsilon > 0$ such that if $|x - y| < \varepsilon$, then $y \in A$.

Show that if A is a bounded subset of $\mathbb{R}$ with property (N), then $\sup A \notin A$.

Solution. We proceed by contradiction. Assume that $z = \sup A$, $z \in A$, and A has property (N). Then since A has property (N) and $z \in A$, there exists some $\varepsilon > 0$ such that if $|x - z| < \varepsilon$, $x \in A$. Setting $x = z + \varepsilon/2$, we see that $|x - z| < \varepsilon$. so $z + \varepsilon/2 \in A$. This contradicts that z is an upper bound of A. Hence $\sup A \notin A$. □

Problem 3. Show that if (a_n) is increasing, then for every $N \in \mathbb{N}$, if $n \geq N$, $a_n \geq a_N$.

Solution. Fix $N \in \mathbb{N}$. We will show if $n \geq N$, then $a_n \geq a_N$ by induction. The base case is $n = N$, where $a_n \geq a_N$ is easily verified. Assume that $a_n \geq a_N$. Then $a_{n+1} \geq a_n$ by the increasing assumption, so $a_{n+1} \geq a_n \geq a_N$. Hence, the result follows by induction. □

Problem 4. Assume (a_n) is increasing. Show that (a_n) is not bounded above if and only if (a_n) diverges to ∞.

Solution. We first show that if (a_n) is not bounded, then it diverges to ∞. Fix $M \in \mathbb{R}$. Since a_n is not bounded, there exists $N \in \mathbb{N}$ such that $a_N \geq M$. Then if $n \geq N$, $a_n \geq a_N$ by the previous problem. It follows that $a_n \geq M$. Since M was arbitrary, (a_n) diverges to ∞.

Now, assume that (a_n) diverges to ∞. Then for every $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, $a_n \geq M$. In particular, $a_N \geq M$. Therefore, since M was arbitrary, (a_n) is unbounded. $\square$